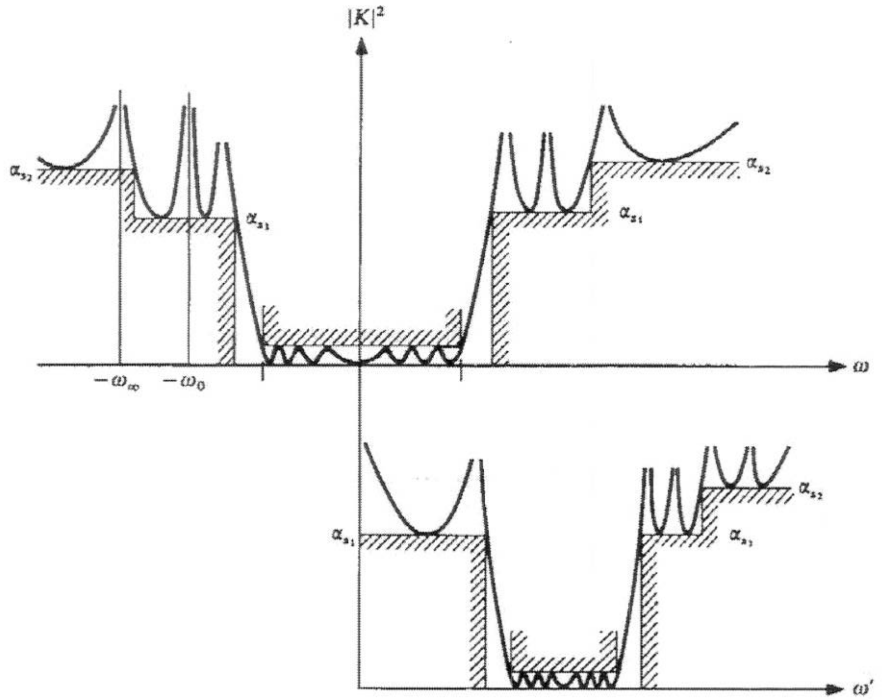


Figure 12-29 Transformation of low-pass to asymmetrical-bandpass filter using Moebius mapping.

The mapping accomplishing this transformation is obtainable in the same way as that of (12-122). The result is

$$\omega'^2 = \left[\omega_p^2 \frac{\omega_p + \omega^{(p)}}{\omega_p + \omega^{(r)}}\right] \frac{\omega + \omega^{(r)}}{\omega + \omega^{(p)}} \qquad (12\text{-}124)$$

which clearly has the required properties.

The mapping of Eq. (12-123) can also be used, within limits, to obtain a frequency-asymmetrical bandpass filter response from a low-pass prototype response. An example is shown in Fig. 12-29, where the choice of ω_0 and ω_∞ is also illustrated. The details of the calculations are left as an exercise (Prob. 12-38).

12-6 LOW-PASS FILTERS WITH MAXIMALLY FLAT DELAY†

Often, a two-port is required to pass a time signal $v_1(t)$ from its input to its output without serious distortion. This requires that all sine-wave components of the signal be treated approximately the same way by the two-port. Thus, let $A \sin(\omega t + \varphi)$ be a signal component at the input, and let the corresponding output signal component be $kA \sin(\omega t - \beta + \varphi)$. Assume next that k is a

† The approach used in this section is somewhat unconventional. It is based on some useful discussions between one of the authors and H. J. Orchard. The usual exposition of this topic can be found in Ref. 9. Reference 12 gives a discussion similar to ours but applied to bandpass filters.

constant, independent of ω, and that the phase shift β equals ωT, where T is also a frequency-independent constant. Then the output component is $kA \sin [\omega(t - T) + \varphi]$: it is thus obtained by scaling the amplitude of the corresponding input component by k and replacing t by $t - T$. The latter operation corresponds to a delay of T seconds between output and input. Since all input components are thus scaled by the same value k and delayed by the same time interval T, obviously the output signal itself will be $kv_1(t - T)$, that is, a scaled and delayed (but undistorted) replica of the input.

As discussed in Sec. 6-2, the phase lag between the output and input voltages of a doubly terminated two-port is $\beta = \angle H(j\omega)$. The argument just completed thus indicates that for distortionless signal transmission the condition

$$\beta = \angle H(j\omega) = \omega T \tag{12-125}$$

must hold in the frequency range of the signal.† Thus, the *phase must be a linear function of* ω. Equivalently, the *phase delay*

$$T_p(\omega) \triangleq \frac{\beta(\omega)}{\omega} \tag{12-126}$$

and the *group delay* (also often called *envelope delay* or *differential delay*)

$$T_g(\omega) \triangleq \frac{d\beta(\omega)}{d\omega} \tag{12-127}$$

must be *constant* in the frequency range of the signal.

Note that $T_p(\omega)$ is the actual time displacement between the input and output signals for a steady-state sine wave of frequency ω; $T_g(\omega)$, on the other hand, does not have such direct physical meaning for low-pass circuits.‡ However, the variations of $T_g(\omega)$ provide a sensitive measure for the departures of $\beta(\omega)$ from the ideal linear characteristics around the frequency ω. Furthermore, if the input signal is a sine wave of frequency ω, modulated by a second sine wave of much lower frequency ω_L, then after demodulation the phase delay of the low-frequency signal is $[\beta(\omega + \omega_L) - \beta(\omega)]/\omega_L \approx T_g(\omega)$. Thus, $T_g(\omega)$ can readily be measured by measuring the phase delay of the demodulated signal. For these reasons, $T_g(\omega)$ is often used in design.

The calculation of $T_g(\omega)$ can proceed from Eq. (6-7)

$$\beta = \angle H(j\omega) = \text{Im}\,[\ln H(j\omega)] \tag{12-128}$$

By (12-127)

$$T_g = \frac{d\beta}{d\omega} = \text{Im}\left[\frac{d}{d\omega}\ln H(j\omega)\right] = \text{Im}\left[j\frac{d}{d(j\omega)}\ln H(j\omega)\right] = \text{Re}\left[\frac{d}{ds}\ln H(s)\right]_{s=j\omega} \tag{12-129}$$

† In addition, of course, α must be a constant in the same range to assure that the scale factor k is also constant.

‡ Although it is possible to find physical interpretation of $T_g(\omega)$ in narrow-band frequency-multiplex systems.

Since the even (odd) part of a rational function of s is real (imaginary) for $s = j\omega$, it follows that

$$T_g(\omega) = \text{Re}\left[\frac{dH(s)/ds}{H(s)}\right]_{s=j\omega} = \left\{\text{Ev}\left[\frac{1}{H(s)}\frac{dH(s)}{ds}\right]\right\}_{s=j\omega} \qquad (12\text{-}130)$$

Here, as before, Ev $[f(s)]$ denotes the even part of $f(s)$.

Assume now a polynomial transducer factor $H(s) \equiv E(s)$. Then (12-130) gives

$$T_g(\omega) = \frac{1}{2}\left[\frac{E'(s)}{E(s)} + \frac{E'(-s)}{E(-s)}\right]_{s=j\omega} \qquad (12\text{-}131)$$

where $E'(s) \triangleq dE(s)/ds$, and $E'(-s)$ is obtained by substituting $-s$ for s in $E'(s)$. Evidently, we can define $F(s)$, an even rational function of s such that for $s = j\omega$, $F(j\omega) = T_g(\omega)$. From (12-131)

$$F(s) = \frac{1}{2}\left[\frac{E'(s)}{E(s)} + \frac{E'(-s)}{E(-s)}\right] = \frac{E'(s)E(-s) + E'(-s)E(s)}{2E(s)E(-s)} \qquad (12\text{-}132)$$

Assume now that the input signal contains most of its energy in the low-frequency region. Then, $T_g(\omega)$ should be constant around $\omega = 0$. Using maximally flat approximation, therefore, we require that the conditions

$$\begin{aligned} T_g(\omega) &= T \\ \frac{dT_g(\omega)}{d(\omega^2)} &= 0 \\ \frac{d^2T_g(\omega)}{d(\omega^2)^2} &= 0 \\ &\cdots\cdots\cdots\cdots \\ \frac{d^{(n-1)}T_g(\omega)}{d(\omega^2)^{n-1}} &= 0 \end{aligned} \qquad (12\text{-}133)$$

hold for $\omega = 0$, where n is the degree of $E(s)$. These conditions are analogous to (12-11). Note that since $T_g(\omega)$ is an *even* rational function of ω, only the derivatives with respect to ω^2 need to be included in (12-133). In terms of s, (12-133) becomes

$$F(s) = T \qquad \frac{d^kF(s)}{d(s^2)^k} = 0 \qquad k = 1, 2, \ldots, n-1 \qquad (12\text{-}134)$$

for $s = 0$.

We shall now use Eqs. (12-132) and (12-134) to derive the maximally flat delay polynomial $E(s)$. As (12-132) shows, the denominator of $F(s)$ is $2E(s)E(-s)$. Hence, we shall assume a solution of (12-134) in the form

$$F(s) = T + \frac{P(s)}{2E(s)E(-s)} \qquad (12\text{-}135)$$

where $P(s)$ is some even polynomial. If the degree of $E(s)$ is n, then that of $P(s)$ is, by (12-133) and (12-135), at most $2n$; hence we can write

$$P(s) = p_0 + p_1 s^2 + \cdots + p_n s^{2n} \tag{12-136}$$

By (12-134), in close analogy to (12-12), we obtain then

$$p_0 = p_1 = p_2 = \cdots = p_{n-1} = 0 \tag{12-137}$$

Hence, from (12-135),

$$F(s) = T + \frac{p_n s^{2n}}{2E(s)E(-s)} = \frac{(2T)E(s)E(-s) + p_n s^{2n}}{2E(s)E(-s)} \tag{12-138}$$

But as (12-132) shows, the numerator polynomial of $F(s)$ is only of degree $2n - 1$. Hence, the coefficient of s^{2n} in the numerator on the right-hand side of (12-138) must vanish. Therefore, if $E(s)$ is in the form

$$E(s) = \sum_{i=0}^{n} a_i s^i \tag{12-139}$$

the cancellation of the highest-order coefficient in (12-138) requires

$$2Ta_n^2(-1)^n + p_n = 0 \qquad p_n = (-1)^{n+1} 2Ta_n^2 \tag{12-140}$$

Now any constant factor can be associated with $E(s)$ without affecting the group delay, as is evident, for example, from (12-131). Hence we choose at this stage $a_n = 1$. Furthermore, we normalize the time scale such that $T = 1$. This implies, by Eq. (1-24) of Sec. 1-4, that the ω and f values are scaled by $1/T$; the frequency and radian frequency units are thus both equal to $\omega_0 = 1/T$.

With these assumptions (12-140) gives $p_n = (-1)^{n+1}2$ and hence, by Eqs. (12-132) and (12-138),

$$\begin{aligned} E'(s)E(-s) + E'(-s)E(s) &= 2E(s)E(-s) + 2(-1)^{n+1}s^{2n} \\ s^{-2n}[E'(s)E(-s) + E'(-s)E(s) - 2E(s)E(-s)] &= 2(-1)^{n+1} \\ s^{-2n}\,\mathrm{Ev}\,\{[E'(s) - E(s)]E(-s)\} &= (-1)^{n+1} \end{aligned} \tag{12-141}$$

Differentiating both sides of the last equation with respect to s and noting that

$$\frac{d}{ds}\,\mathrm{Ev}\,f(s) = \mathrm{Od}\,\frac{df(s)}{ds} \tag{12-142}$$

and

$$\frac{dE(-s)}{ds} = -\frac{dE(-s)}{d(-s)} = -E'(-s) \tag{12-143}$$

we obtain after a simple calculation

$$\mathrm{Ev}\,\{[sE''(s) - 2(n+s)E'(s) + 2nE(s)]E(-s)\} = 0 \tag{12-144}$$

Let us denote the polynomial in the square brackets by $D(s)$:

$$D(s) \triangleq sE''(s) - 2(n+s)E'(s) + 2nE(s). \tag{12-145}$$

Note that by (12-139) the degree of $D(s)$ appears to be n; this, however, is an illusion, since the coefficient of s^n in $D(s)$ is, by (12-145),

$$d_n = -2na_n + 2na_n \equiv 0 \tag{12-146}$$

Thus, $D(s)$ is only of degree $n-1$.

By (12-144), Ev $[D(s)E(-s)] \equiv 0$, so that $D(s)E(-s)$ must be a *pure odd* real polynomial. Hence, its zeros must occur in quadrantal symmetry. Furthermore, as discussed in Sec. 6-2, realizability requires $E(s)$ to be a strictly Hurwitz polynomial; hence all n zeros of $E(-s)$ must lie in the RHP. Consequently, $D(s)$ must contain n LHP zeros to complete the symmetric pattern. However, as shown above, $D(s)$ is only of degree $n-1$.

The only possible escape out of this contradiction is to assume either that $E(s) \equiv 0$ or that $D(s) \equiv 0$. The former leads to a useless result; hence, we must set

$$D(s) \triangleq sE''(s) - 2(n+s)E'(s) + 2nE(s) \equiv 0 \tag{12-147}$$

This second-order differential equation forms the basis of the solution of our problem. Substituting (12-139) into (12-147) gives the relation

$$s\sum_{i=2}^{n} i(i-1)a_i s^{i-2} - 2(n+s)\sum_{i=1}^{n} ia_i s^{i-1} + 2n\sum_{i=0}^{n} a_i s^i \equiv 0 \tag{12-148}$$

Hence, the coefficient of s^k $(0 \le k \le n-1)$ satisfies†

$$(k+1)ka_{k+1} - 2n(k+1)a_{k+1} - 2ka_k + 2na_k = 0$$

$$a_k = \frac{(k+1)(2n-k)}{2(n-k)}a_{k+1} \tag{12-149}$$

Since we have already set $a_n = 1$, the remaining coefficients are given by

$$\begin{aligned}
a_{n-1} &= \frac{n(n+1)}{(1)(2)}a_n = \frac{n(n+1)}{(2^1)(1!)} \\
a_{n-2} &= \frac{(n-1)(n+2)}{(2)(2)}a_{n-1} = \frac{(n-1)n(n+1)(n+2)}{(2^2)(2!)} \\
a_{n-3} &= \frac{(n-2)(n+3)}{(2)(3)}a_{n-2} = \frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{(2^3)(3!)} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{12-150}$$

The coefficient of s^{n-k} is thus clearly

$$a_{n-k} = \frac{(n-k+1)(n-k+2)\cdots n(n+1)\cdots(n+k)}{2^k k!} \tag{12-151}$$

† Notice that (12-149) holds also for $k = n$ in a trivial way since $a_{n+1} \equiv 0$.

or, using $i = n - k$ and indicating the order of $E(s)$ as a superscript of a_i,

$$a_i^n = \frac{(i+1)(i+2)\cdots(2n-i)}{2^{n-i}(n-i)!} = \frac{(2n-i)!}{2^{n-i}i!\,(n-i)!} \qquad (12\text{-}152)$$

The coefficients calculated from (12-152) are given, for $1 \le n \le 11$, in Table 12-9. The polynomials defined by these coefficients are closely related to the Bessel polynomials. Since they were first described by W. E. Thomson of the British Post Office Research Station, the filters derived from these transfer functions are called *Bessel filters, Thomson filters,* or (as a compromise) Bessel-Thomson filters.

By the construction of the formula for a_i^n, it is clear that under the stated assumptions [polynomial transfer function, strictly Hurwitz $E(s)$] the derived solution to the maximally flat delay approximation problem is *unique*.

Next, rewrite the formula (12-152) for a_i^n as follows

$$\begin{aligned} a_i^n &= \frac{(2n-i-2)!}{2^{n-i-1}(i-2)!\,(n-i-1)!}\,\frac{(2n-i)(2n-i-1)}{2i(i-1)(n-i)} \\ &= \frac{(2n-i-2)!}{2^{n-i-1}(i-2)!\,(n-i-1)!}\,\frac{2(n-i)(2n-1)+i(i-1)}{2i(i-1)(n-i)} \\ &= (2n-1)\frac{(2n-i-2)!}{2^{n-i-1}i!\,(n-i-1)!} + \frac{(2n-i-2)!}{2^{n-i}(i-2)!\,(n-i)!} \qquad (12\text{-}153) \end{aligned}$$

Using (12-152), with n replaced first by $n-1$ and then by $n-2$, (12-153) gives

$$a_i^n = (2n-1)a_i^{n-1} + a_{i-2}^{n-2} \qquad (12\text{-}154)$$

It is easy to show (see Prob. 12-41) that (12-154) holds for all i between 0 and n, that is, for $0 \le i \le n$. Hence, the polynomials of degrees n, $n-1$, and $n-2$ satisfy

$$\begin{aligned} E^n(s) &= \sum_{i=0}^{n} a_i^n s^i = (2n-1)\sum_{i=0}^{n-1} a_i^{n-1}s^i + s^2\sum_{i=0}^{n} a_{i-2}^{n-2}s^{i-2} \\ &= (2n-1)E^{n-1}(s) + s^2E^{n-2}(s) \qquad (12\text{-}155) \end{aligned}$$

Here, we indicated the degree of $E(s)$ by its superscript.

Equations (12-154) and (12-155) provide a useful recurrence process for calculating the $E^n(s)$. For $n = 0$, $a_0^0 = a_n^0 = 1$; for $n = 1$, $a_1^1 = a_n^1 = 1$; and from (12-152), $a_0^1 = 2!/[(2)(0!\,1!)] = 1$ since $0! = 1! = 1$. Hence $E^0(s) = 1$, $E^1(s) = s + 1$ and therefore by (12-155)

$$E^2(s) = 3E^1(s) + s^2E^0(s) = s^2 + 3s + 3$$

$$E^3(s) = 5E^2(s) + s^2E^1(s) = s^3 + 6s^2 + 15s + 15$$

etc. Thus, all entries of Table 12-9 can be obtained.

Another useful relation obtainable from (12-152) concerns the *derivative* of $E^n(s)$:

$$\frac{dE^n(s)}{ds} = \sum_{i=1}^{n} a_i^n i s^{i-1} = \sum_{k=0}^{n-1} b_k^n s^k \qquad (12\text{-}156)$$

Table 12-9 Coefficients of the maximally flat-delay polynomial $E^n(s)$†

Note that $a_n(s)$ is chosen as 1 for all n

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
1	1										
2	3	3									
3	15	15	6								
4	105	105	45	10							
5	945	945	420	105	15						
6	10,395	10,395	4,725	1,260	210	21					
7	135,135	135,135	62,370	17,325	3,150	378	28				
8	2,027,025	2,027,025	945,945	270,270	51,975	6,930	630	36			
9	34,459,425	34,459,425	16,216,200	4,729,752	945,945	135,135	13,860	990	45		
10	654,729,075	654,729,075	310,134,825	91,891,800	18,918,900	2,837,835	315,315	25,740	1,485	55	
11	13,749,310,575	13,749,310,575	6,547,290,750	1,964,187,225	413,513,100	64,324,260	7,567,560	675,675	45,045	2,145	66

† Reproduced by permission from L. Weinberg, "Network Analysis and Synthesis," p. 500, McGraw-Hill, New York, 1962; reprinted by Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1975.

Table 12-10 Zeros (natural modes) for maximally flat-delay filters†

n	
1	-1.0000000
2	$-1.5000000 \pm j0.8660254$
3	$-2.3221854;\ -1.8389073 \pm j1.7543810$
4	$-2.8962106 \pm j0.8672341;\ -2.1037894 \pm j2.6574180$
5	$-3.6467386;\ -3.3519564 \pm j1.7426614;\ -2.3246743 \pm j3.5710229$
6	$-4.2483594 \pm j0.8675097;\ -3.7357084 \pm j2.6262723;\ -2.5159322 \pm j4.4926730$
7	$-4.9717869;\ -4.7582905 \pm j1.7392861;\ -4.0701392 \pm j3.5171740;\ -2.6856769 \pm j5.4206941$
8	$-5.5878860 \pm j0.8676144;\ -2.8389840 \pm j6.3539113;\ -4.3682892 \pm j4.4144425;\ -5.2048408 \pm j2.6161751$
9	$-6.2970193;\ -6.1293679 \pm j1.7378484;\ -5.6044218 \pm j3.4981573;\ -4.6384399 \pm j5.3172717;\ -2.9792608 \pm j7.291637$
10	$-6.9220449 \pm j0.8676651;\ -3.1089162 \pm j8.2326995;\ -6.6152916 \pm j2.6115683;\ -5.9675282 \pm j4.3849471;\ -4.886195 \pm j6.2249855$
11	$-7.6223398;\ -6.3013375 \pm j5.2761917;\ -5.1156483 \pm j7.1370208;\ -7.4842299 \pm j1.737028;\ -7.0578924 \pm j3.489145;$ $-3.2297221 \pm j9.1771116$

† Reproduced by permission from L. Weinberg, "Network Analysis and Synthesis," p. 500, McGraw-Hill, New York, 1962; reprinted by Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1975.

where $k \triangleq i - 1$. Clearly, from (12-152),

$$b_k^n = (k+1)a_{k+1}^n = (k+1)\frac{(2n-k-1)!}{2^{n-k-1}(k+1)!(n-k-1)!}$$
$$= \frac{(2n-k)!}{2^{n-k}k!(n-k)!} - \frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!} \qquad (12\text{-}157)$$

Using (12-152) again to identify the two terms on the right-hand side, we find

$$b_k^n = a_k^n - a_{k-1}^{n-1} \qquad (12\text{-}158)$$

For $k = 0$, $a_{k-1}^{n-1} \equiv 0$, and hence (12-158) gives $b_0^n = a_0^n$, which is valid since $b_0^n = a_1^n = a_0^n$ for all n. Similarly, for $k = n$, (12-158) predicts $b_n^n = a_n^n - a_{n-1}^{n-1} = 1 - 1 \equiv 0$ for all n which is also true. Hence, (12-158) holds for $0 \le k \le n$ and thus

$$\frac{dE^n(s)}{ds} = \sum_{k=0}^{n} a_k^n s^k - s\sum_{k=0}^{n} a_{k-1}^{n-1} s^{k-1} = E^n(s) - sE^{n-1}(s) \qquad (12\text{-}159)$$

Equations (12-155) and (12-159) can be used (see Probs. 12-43 to 12-45) to show that $E^n(s)$ does indeed satisfy (12-141) and thus the maximally flat-delay property. Using these relations, it can also be readily shown† that all $E^n(s)$ are strictly Hurwitz, as required by physical realizability and as assumed earlier in the derivation. Dividing both sides of (12-155) by $sE^{n-1}(s)$ gives

$$\frac{E^n(s)}{sE^{n-1}(s)} = \frac{2n-1}{s} + \frac{sE^{n-2}(s)}{E^{n-1}(s)} \qquad (12\text{-}160)$$

For $n = 2$, $E^{n-2}(s) \equiv 1$, and $E^{n-1}(s) = s + 1$, as shown earlier. Hence,

$$\frac{E^2(s)}{sE^1(s)} = \frac{3}{s} + \frac{s}{s+1} \qquad (12\text{-}161)$$

is a PR function. So is its reciprocal, and hence so is

$$\frac{E^3(s)}{sE^2(s)} = \frac{5}{s} + \frac{sE^1(s)}{E^2(s)} \qquad (12\text{-}162)$$

Thus, by (12-160), $E^4(s)/sE^3(s)$ is also PR. Proceeding this way, we see that $E^n(s)/sE^{n-1}(s)$ is PR for all n. Hence, $E^n(s)$ and $E^{n-1}(s)$ both have at least the Hurwitz property for all n. To show that they are *strictly* Hurwitz we must prove that they do not have any $j\omega$-axis zeros, i.e., any factors of the forms $s^2 + \omega_0^2$ or s. Such factors (as shown in Chap. 4) are present in both the even and odd parts of, say, $E^{n-1}(s)$ and therefore of $sE^{n-1}(s)/E^n(s)$.

By (12-159),

$$\frac{dE^n(s)/ds}{E^n(s)} = 1 - \frac{sE^{n-1}(s)}{E^n(s)} \qquad (12\text{-}163)$$

† This derivation is due to H. J. Orchard.

and hence, from Eqs. (12-132) and (12-138) with $T = 1$,

$$F(s) = \text{Ev } \frac{dE^n(s)/ds}{E^n(s)} = 1 - \text{Ev } \frac{sE^{n-1}(s)}{E^n(s)} = 1 + \frac{p_n s^{2n}}{2E^n(s)E^n(-s)}$$
$$\text{Ev } \frac{sE^{n-1}(s)}{E^n(s)} = \frac{-p_n s^{2n}}{2E^n(s)E^n(-s)} \tag{12-164}$$

This shows that $j\omega$-axis zeros of $E^{n-1}(s)$ can only occur for $s = 0$. However, by (12-152), the constant term of $E^{n-1}(s)$ is

$$a_0^{n-1} = \frac{(2n-2)!}{2^{n-1}(n-1)!} \neq 0 \tag{12-165}$$

and hence $E^{n-1}(s)$ is not zero for $s = 0$. Thus, $E^{n-1}(s)$ has *no* $j\omega$-axis zeros and is therefore strictly Hurwitz. Since our derivation is valid for all n, the strictly Hurwitz character of the maximally flat-delay polynomials has thus been proved.†

It is also reassuring to note that, by (12-152),

$$\frac{a_i^n}{a_0^n} = \frac{(2n-i)!}{2^{n-i}i!\,(n-i)!}\,\frac{2^n n!}{(2n)!} = \frac{n!/(n-i)!}{2^{-i}(2n)!/(2n-i)!}\,\frac{1}{i!}$$
$$= \frac{(n-i+1)(n-i+2)\cdots(n-1)n}{2^{-i}(2n-i+1)(2n-i+2)\cdots(2n-1)2n}\,\frac{1}{i!} \tag{12-166}$$

and hence for $n \to \infty$

$$\frac{a_i^n}{a_0^n} \to \frac{n^i}{2^{-i}(2n)^i}\,\frac{1}{i!} = \frac{1}{i!} \tag{12-167}$$

Hence

$$E^n(s) = a_0^n \sum_{i=0}^{n} \frac{a_i^n}{a_0^n} s^i \to a_0^n \sum_{i=0}^{\infty} \frac{s^i}{i!} \qquad E^n(s) \to a_0^n e^s \tag{12-168}$$

that is, $E^n(s)$ tends (apart from the scale factor a_0^n) to the unit delay operator e^s.

The asymptotic behavior of the loss $\alpha(\omega) = 10 \log [E(j\omega)E(-j\omega)]$ corresponding to $E^n(s)$ can also be found for large values of n. We have

$$E^n(s)E^n(-s) = \left(\sum_{i=0}^{n} a_i^n s^i\right)\left[\sum_{i=0}^{n} a_i^n(-1)^i s^i\right] \tag{12-169}$$

† The alert reader will have noted that the above proof of the strictly Hurwitz property of $E^n(s)$ does not exclude the possibility of a *common* zero of $E^n(s)$ and $E^{n-1}(s)$ in the RHP or on the $j\omega$ axis. It can, however, be easily shown that $E^n(s)$ and $E^{n-1}(s)$ cannot have any zeros in common. The proof is based on (12-155) and is hinted at in Prob. 12-49.

This gives a pure even polynomial of the form $b_0 + b_1 s^2 + b_2 s^4 + \cdots$. To find the first few coefficients, we obtain, by repeatedly using (12-149),

$$a_1^n = a_0^n$$
$$a_2^n = \frac{n-1}{2n-1} a_0^n$$
$$a_3^n = \frac{n-2}{3(2n-1)} a_0^n \qquad (12\text{-}170)$$
$$a_4^n = \frac{(n-2)(n-3)}{6(2n-1)(2n-3)} a_0^n$$

Hence, the lowest-order coefficients of $E^n(s)E^n(-s)$ are

$$b_0 = (a_0^n)^2$$
$$b_1 = 2a_0^n a_2^n - (a_1^n)^2 = -\frac{(a_0^n)^2}{2n-1} \qquad (12\text{-}171)$$
$$b_2 = 2a_0^n a_4^n - 2a_1^n a_3^n + (a_2^n)^2 = \frac{n-1}{2n-3}\frac{(a_0^n)^2}{(2n-1)^2}$$

Thus for $s = j\omega$, that is, for $s^2 = -\omega^2$,

$$E^n(j\omega)E^n(-j\omega) = (a_0^n)^2\left[1 + \frac{\omega^2}{2n-1} + \frac{n-1}{2n-3}\left(\frac{\omega^2}{2n-1}\right)^2 + \cdots\right] \qquad (12\text{-}172)$$

It is of interest to compare this expression with the Taylor series of the *gaussian function*†

$$(a_0^n)^2 e^{\omega^2/(2n-1)} = (a_0^n)^2\left[1 + \frac{\omega^2}{2n-1} + \frac{1}{2}\left(\frac{\omega^2}{2n-1}\right)^2 + \cdots\right] \qquad (12\text{-}173)$$

Clearly, the relative difference is

$$\frac{(a_0^n)^2 e^{\omega^2/(2n-1)} - E^n(j\omega)E^n(-j\omega)}{(a_0^n)^2 e^{\omega^2/(2n-1)}} = e^{-\omega^2/(2n-1)}\left[\frac{-\omega^4}{2(2n-3)(2n-1)^2} + \text{higher-order terms}\right] \qquad (12\text{-}174)$$

Hence, for $n \geq 3$, and for ω values not in excess of, say, 3,‡ the approximation

$$E^n(j\omega)E^n(-j\omega) \approx (a_0^n)^2 e^{\omega^2/(2n-1)} \qquad (12\text{-}175)$$

holds. The loss is then

$$\alpha = 10 \log\left[E^n(j\omega)E^n(-j\omega)\right] \approx 20 \log a_0^n + \frac{10 \log e}{2n-1}\omega^2 \qquad (12\text{-}176)$$

† A gaussian function $f(x)$ is of the form $f(x) = Ae^{-Bx^2}$.

‡ In terms of denormalized units, this condition is $\omega < 3/T$.

The approximation of (12-175) can also be made in (12-138). Using (12-140), we get then (with $T = 1$)

$$T_g(j\omega) = 1 - \frac{\omega^{2n}}{E(j\omega)E(-j\omega)} = 1 - (a_0^n)^{-2}\omega^{2n}e^{-\omega^2/(2n-1)} \qquad (12\text{-}177)$$

Often, we require the zero-frequency loss to equal zero.† If we choose $H(s) = E^n(s)/a_0^n$, this will be achieved. Then (12-176) will be replaced by

$$\alpha \approx \frac{10 \log e}{2n-1}\omega^2 \qquad (12\text{-}178)$$

The group delay will, of course, remain unchanged.

† This permits $R_G = R_L$ in the resulting circuit.

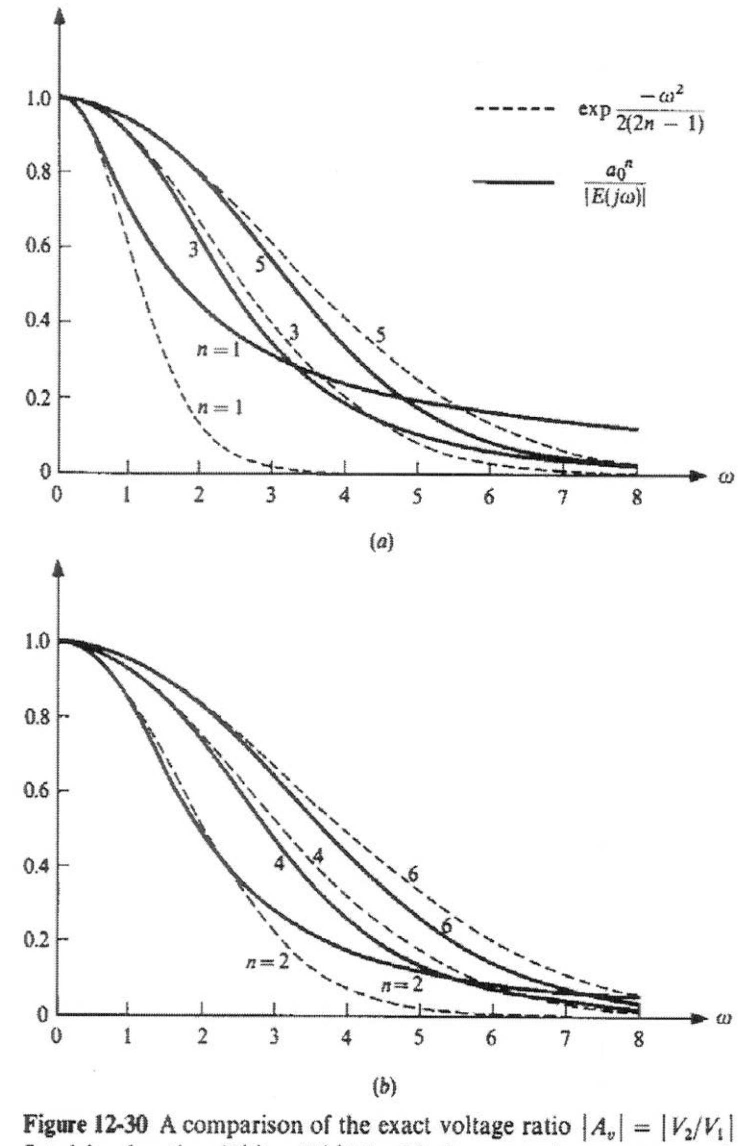
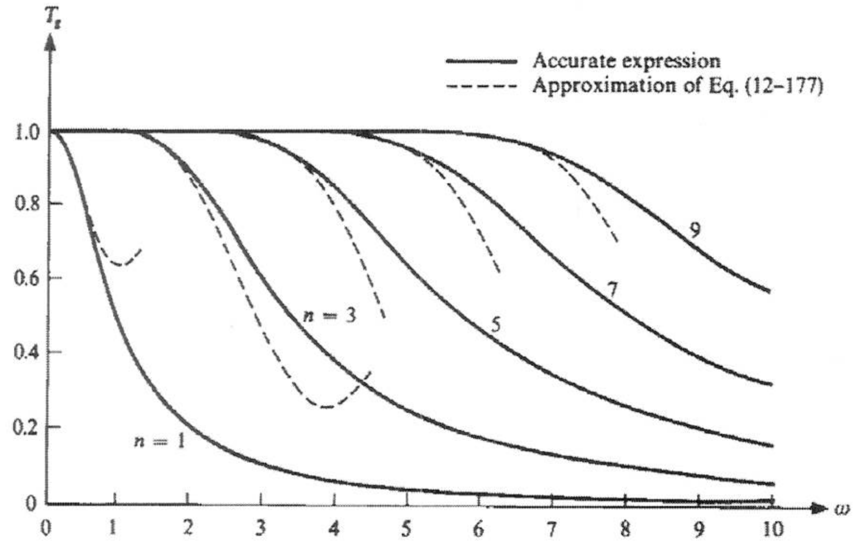
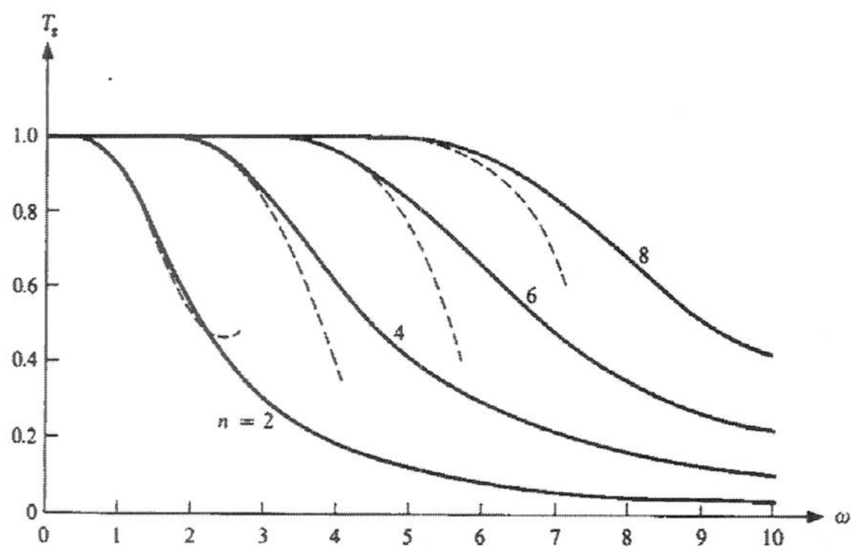


Figure 12-30 A comparison of the exact voltage ratio $|A_v| = |V_2/V_1|$ calculated from the maximally flat delay function $H(s) = E^n(s)/a_0^n$ with the approximate response found from Eq. (12-175); (*a*) for odd degrees n; (*b*) for even degrees.



(a)



(b)

Figure 12-31 A comparison of the exact group delay $T_g(j\omega)$ calculated from the maximally flat delay transfer function $H(s) = E^n(s)/a_0^n$ with the response obtained from the approximation (12-177) (*a*) for odd degrees n; (*b*) for even degrees.

The output-input amplitude response $a_0^n/|E(j\omega)|$ and the group-delay response $T_g(j\omega)$ associated with $H(s) = E^n(s)/a_0^n$ are plotted for low values of n in Figs. 12-30 and 12-31, respectively. The maximally flat property is evident for the delay response. To illustrate the quality of the approximations given in Eqs. (12-175) and (12-177), these figures also show the responses computed from the approximate expressions.

Example 12-11 Find a transfer function such that the group delay is 10 μs at low frequencies, with a deviation less than 0.5 percent for $|f| \leq 30$ kHz. The loss should be zero at zero frequency and should remain less than 1 dB for $|f| \leq 30$ kHz.

To satisfy the zero-loss condition for $\omega = 0$, we choose the frequency-normalized transfer function in the form

$$H(s) = \frac{E^n(s)}{a_0^n}$$

The degree n can be found from the specifications on the loss and delay deviations. Since $\omega T = (2\pi 3 \times 10^4)(10^{-5}) \approx 1.885$, with (12-178) the loss requirement gives at the normalized limit frequency ωT

$$\frac{10 \log e}{2n - 1}(\omega T)^2 \approx \frac{4.343}{2n - 1}[(2\pi 3 \times 10^4)(10^{-5})]^2 < 1$$

This leads to

$$n > 8.2$$

Hence, $n = 9$ is the minimum usable degree. For $n = 9$, the relative delay deviation at the limit frequency is, by (12-177),

$$\frac{\Delta T}{T} \approx (a_0^9)^{-2}(\omega T)^{18} e^{-(\omega T)^2/17} \approx 6.17 \times 10^{-11}$$

This is much less than the prescribed 0.5 percent and hence meets the delay specification.

Using Table 12-9, we find frequency-normalized transfer function to be

$$\begin{aligned} H(s) = \frac{E^9(s)}{a_0^9} &= \frac{1}{a_0^9} \sum_{i=0}^{9} a_i^9 s^i \\ &\approx 2.90196 \times 10^{-8} s^9 + 1.30588 \times 10^{-6} s^8 \\ &\quad + 2.87294 \times 10^{-5} s^7 + 4.02212 \times 10^{-4} s^6 \\ &\quad + 3.92157 \times 10^{-3} s^5 + 2.74510 \times 10^{-2} s^4 \\ &\quad + 0.137256 s^3 + 0.470588 s^2 + s + 1 \end{aligned}$$

To denormalize $H(s)$, s must be replaced by sT. This changes a_i^n into $a_i^n T^i$; in our example the leading coefficient becomes 2.90196×10^{-53}. Hence normalized calculation is advisable to avoid over- or underflows.

It is interesting to note that the output-input transfer function $1/|H(j\omega)|$ and the impulse response of the maximally flat-delay filter are both gaussian functions of ω and t, respectively. The proof is outlined in Prob. 12-48.

As mentioned earlier, the Bessel-Thomson (maximally flat-delay) approximation is analogous to the Butterworth (maximally flat-loss) approximation discussed in Sec. 12-2. It can be anticipated that a more efficient function may be obtained if an equal-ripple, rather than a maximally flat, approximation to a constant delay is achieved. Such an approximation leads to a function analogous to the Chebyshev function described in Sec. 12-3. It is illustrated in Fig. 12-32. The derivation of such functions can be performed using the Z variable in-